Control Engineering 2014-2015
Exam 3 November 2014
Prof. C. De Persis

- You have **3 hours** to complete the exam.
- You **can** use books and notes but not smart phones, computers, tablets and the like.
- For each problem, please do your best to write your answers within the box that is below the problem text. Notice that the box extends over the page following the one where the problem is stated.
- There are questions/exercises labeled as **Bonus**. These questions/exercises are optional and give you **extra** points if answered correctly.
- Please write down your Surname, Name, Student ID on each sheet.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.
- At the end of the exam, please fill in the course evaluation form. Thank you.

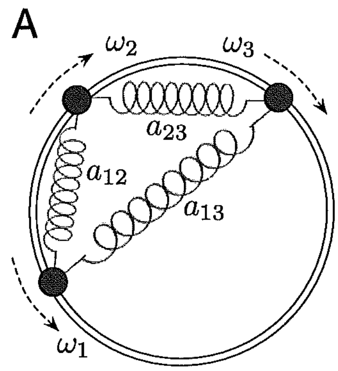

Figure 1: From Dörfler et al. Synchronization in complex oscillator networks and smart grids. PNAS 2013.

1. [15pts] Consider two masses evolving on a unitary circle (radius=1) and connected by a linear spring with coefficient a_{12} (see Figure 1, where you should neglect the third mass (labeled as ω_3), and the springs labeled as a_{13}, a_{23}). A torque τ_i is applied to mass i, $i = 1, 2$, resulting in a circular motion of the mass. The variables φ_1, φ_2 are the angular displacements, ω_1, ω_2, the resulting angular velocities and J_1, J_2 the moment of inertia of the two rotating masses. Gravity and friction are neglected in this exercise.

   (a) [2pts] Determine the generalized displacement vector q.

   (b) [2pts] Determine the kinetic co-energy $T^*(q, \dot{q})$ of the system (this is the kinetic co-energy of the two rotating masses).

   (c) [2pts] Determine the potential energy $V(q)$ of the system (this is the potential energy stored in the spring).
   Hint The chord of an angle θ is the length of the chord between two points on a unit circle separated by that angle. The chord length crd is given by

   $$\mathrm{crd} = 2\sin\frac{\theta}{2}.$$

   (d) [2pts] Determine the vector F of external generalized forces.

   (e) [1pt] Determine the Lagrangian of the system.

   (f) [6pts] Determine the equations of motion of the system.

   (g) [5pts] **(Bonus)** Determine the equations of motion of the system in Figure 1 considering all the three masses and the three springs.

(a)

$$q = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \dot{q} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

(b)

$$T^*(q,\dot q) = \sum_{i=1}^{2} \frac{1}{2} J_i \omega_i^2 = \sum_{i=1}^{2} \frac{1}{2} J_i \dot q_i^2$$

(c) Spring elongation $2\sin\frac{\psi_1-\psi_2}{2}$; potential energy stored in the spring

$$V(q) = \frac{1}{2} a_{12} \cdot 4\sin^2\frac{\varphi_1-\varphi_2}{2} = 2a_{12}\sin^2\frac{q_1-q_2}{2} \qquad \text{[1.5pts]}$$

Recall that $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$; hence

$$V(q) = 2a_{12}\frac{1-\cos(q_1-q_2)}{2} = a_{12}\left(1-\cos(q_1-q_2)\right) \qquad \text{[0.5pts]}$$

(d)

$$F = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix};$$

(e)

$$L(q,\dot q) = T^*(q,\dot q) - V(q) = \sum_{i=1}^{2} \frac{1}{2} J_i \dot q_i^2 - a_{12}\left(1-\cos(q_1-q_2)\right)$$

(f)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = F_i, \qquad i = 1,2$$

$$\frac{\partial L}{\partial \dot q_i} = J_i \dot q_i; \quad \frac{\partial L}{\partial q_1} = -a_{12}\sin(q_1-q_2); \quad \frac{\partial L}{\partial q_2} = +a_{12}\sin(q_1-q_2);$$

Hence, for $i = 1,2$

$$\begin{cases} J_1\ddot q_1 + a_{12}\sin(q_1-q_2) = \tau_1 \\ J_2\ddot q_2 - a_{12}\sin(q_1-q_2) = \tau_2 \end{cases}$$

<u>Alternative</u> If

$$L(q,\dot q) = \sum_{i=1}^{2} \frac{1}{2} J_i \dot q_i^2 - 2a_{12}\sin^2\frac{q_1-q_2}{2},$$

then $\left(\sin\alpha = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\right)$

$$\frac{\partial L}{\partial q_1} = -a_{12} 2\sin\frac{q_1-q_2}{2}\cos\frac{q_1-q_2}{2}\cdot\frac{1}{2} = -a_{12}\sin(q_1-q_2)$$

(g)

$$q = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \qquad T^*(q,\dot q) = \sum_{i=1}^{3} \frac{1}{2} J_i \dot q_i^2$$

$$\begin{aligned} V(q) &= 2a_{12}\sin^2\frac{q_1-q_2}{2} + 2a_{13}\sin^2\frac{q_1-q_3}{2} + 2a_{23}\sin^2\frac{q_2-q_3}{2} \\ &= a_{12}\left(1-\cos(q_1-q_2)\right) + a_{13}\left(1-\cos(q_1-q_3)\right) + a_{23}\left(1-\cos(q_2-q_3)\right) \end{aligned}$$

$$L(q,\dot{q}) = \sum_{i=1}^{3}\frac{1}{2}J_i\dot{q}_i^2 - \sum_{i=1}^{3}\sum_{j>i} a_{ij}\left(1-\cos(q_i-q_j)\right)$$

$$\frac{\partial L}{\partial \dot{q}_i} = J_i\dot{q}_i, \qquad \begin{aligned} \frac{\partial L}{\partial \dot{q}_1} &= -a_{12}\sin(q_1-q_2) - a_{13}\sin(q_1-q_3) \\ \frac{\partial L}{\partial \dot{q}_2} &= a_{12}\sin(q_1-q_2) - a_{23}\sin(q_2-q_3) \\ \frac{\partial L}{\partial \dot{q}_3} &= a_{13}\sin(q_1-q_3) + a_{23}\sin(q_2-q_3) \end{aligned}$$

$$\begin{cases} J_1\ddot{q}_1 + a_{12}\sin(q_1-q_2) + a_{13}\sin(q_1-q_3) = \tau_1 \\ J_2\ddot{q}_2 - a_{12}\sin(q_1-q_2) + a_{23}\sin(q_2-q_3) = \tau_2 \\ J_3\ddot{q}_3 - a_{13}\sin(q_1-q_3) - a_{23}\sin(q_2-q_3) = \tau_3 \end{cases}$$

2. [15pts] Consider the model of an inverted pendulum as discussed in your textbook (Example 4.4 and Exercise 4.9[1]) but with no friction ($c = 0$)

$$\ddot{\theta} = \sin\theta + u\cos\theta,$$

where θ is the angle between the pendulum and the vertical axis and the control input u is the acceleration of the pivot.
Apply the swing-up control

$$u = K(V_0 - V)\dot{\theta}\cos\theta,$$

where K, V_0 are constant parameters and V is the energy function

$$V(\theta, \dot{\theta}) = \cos\theta - 1 + \frac{1}{2}\dot{\theta}^2$$

(a) [1pt] Write the closed-loop system as a function of the state variables $\theta, \dot{\theta}$, i.e. the system obtained by replacing the expression of u into the equations of the inverted pendulum.

(b) [3pts] Choose the state variables x_1, x_2 and express the system you obtained in the question above as a system of first-order non-linear differential equations

$$\dot{x} = f(x).$$

Write $f(x)$ explicitly.

(c) [2pts] Compute all the equilibrium points x_e.

(d) [3pts] Compute the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ associated with the vector field $f(x)$.

(e) [2pts] Compute the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ at the equilibrium points $x = x_e$. How many dynamic matrices A of the linearized systems do you obtain? Provide an expression for all the matrices.

(f) [4pts] Discuss the stability of the dynamic matrix A corresponding to the equilibrium given by the upright position ($\theta = 0$). Do there exist values of K, V_0 for which the origin of the linearized system is asymptotically stable?

(a)

$$\ddot{\theta} = \sin\theta + k(V_0 - \cos\theta + 1 - \frac{1}{2}\dot{\theta}^2)\dot{\theta}(\cos\theta)^2,$$

(b) $x_1 = \theta$, $x_2 = \dot{\theta}$. [1pt]

$$\dot{x} = \begin{bmatrix} x_2 \\ \sin x_1 + k(V_0 - \cos x_1 + 1 - \frac{1}{2}x_2^2)x_2(\cos x_1)^2 \end{bmatrix} =: f(x) \qquad \text{[2pts]}$$

(c)

$$x_e = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$$

with $n \in \mathbb{Z}$.

[1] The textbook is not needed to solve the exercise.

(d)

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ \cos x_1 - 2k(V_0 + 1 - \frac{x_2^2}{2})x_2 \cos x_1 \sin x_1 + 3kx_2(\cos x_1)^2 \sin x_1 & k(V_0+1)(\cos x_1)^2 - k(\cos x_1)^3 - \frac{3k}{2}x_2^2(\cos x_1)^2 \end{bmatrix}$$

(e) For n even

$$A = \begin{bmatrix} 0 & 1 \\ 1 & k(V_0+1) - k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & kV_0 \end{bmatrix}$$

For n odd

$$A = \begin{bmatrix} 0 & 1 \\ -1 & kV_0 + 2k \end{bmatrix}$$

(f) The matrix A when n is even has always one eigenvalue with positive real part. Hence the equilibrium is unstable.

3. [16pts] Consider a compartmental model for drug administration given by

$$\begin{aligned}\dot{x} &= Ax + Bu = \begin{bmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{bmatrix} x + \begin{bmatrix} b_0 \\ 0 \end{bmatrix} u \\ y &= Cx = \begin{bmatrix} 0 & 1 \end{bmatrix} x\end{aligned}$$

where k_0, k_1, k_2 are parameters, $x_1 \in \mathbb{R}$ is the drug concentration in the blood plasma, $x_2 \in \mathbb{R}$ is is the drug concentration in the tissue, and $u \in \mathbb{R}$ is the flow rate of the drug.

(a) [2pts] Determine the reachability matrix W_r. For what values of the parameters k_0, k_1, k_2 and b_0 is the system reachable?

(b) [2pts] Determine the reachable canonical form of the state space equation.

(c) [2pts] Determine the reachability matrix $\tilde{W}_r$ of the reachable canonical form.

(d) [4pts] Set $k_0 = k_1 = k_2 = 1$ and $b_0 = 1$. Determine the gain matrix K such that the eigenvalues of $A - BK$ are $\{-1, -1\}$. What is the characteristic polynomial corresponding to the eigenvalues $\{-1, -1\}$?

(e) [2pts] Determine the characteristic polynomial of the matrix $A - BK$. Does it coincide with the characteristic polynomial determined in (d)?

(f) [3pts] Determine the value k_r and r in the feedback law

$$u = -Kx + k_r r$$

such that the drug concentration in the tissue converges asymptotically to 1.

(g) [5pts] (**Bonus**) Consider a new dynamical system

$$\begin{aligned}\dot{x} &= Ax + Bu = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned} \tag{1}$$

Design a gain L such that the eigenvalues of $A - LC$ are $\{-1, -1\}$ and use it to design a state estimator (observer) for the system (1).

(a)

$$[1\text{pt}]\ W_r = \begin{pmatrix} b_0 & -b_0(k_0 + k_1) \\ 0 & k_2 b_0 \end{pmatrix} \qquad [1\text{pt}]\ b_0^2 k_2 \neq 0$$

(b)

$$s^2 + (k_0 + k_1 + k_2)s + (k_0 + k_1)k_2 - k_1 k_2 = s^2 + (k_0 + k_1 + k_2)s + k_0 k_2 =: s^2 + a_1 s + a_2$$

Reachable canonical form

$$\dot{z} = \begin{bmatrix} -(k_0 + k_1 + k_2) & -k_0 k_2 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

(c) Reachable matrix for the reachable canonical form

$$\tilde{W}_r = \begin{pmatrix} 1 & -(k_0 + k_1 + k_2) \\ 0 & 1 \end{pmatrix}$$

(d) Desired closed-loop polynomial

$$s^2 + p_1 s + p_2 = s^2 + 2s + 1$$

$$K = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 \end{pmatrix} \tilde{W}_r W_r^{-1} = \begin{pmatrix} -1 & 0 \end{pmatrix} \tilde{W}_r W_r^{-1}$$

$$\begin{aligned} W_r &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad W_r^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ \tilde{W}_r &= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \\ K &= \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \end{pmatrix} \end{aligned}$$

(e)

$$\begin{aligned} A - BK &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

$$p_{ch}(s) = (s+1)^2 \qquad \text{Yes, it does.}$$

(f)

$$\begin{aligned} k_r &= -\frac{1}{C(A-BK)^{-1}B} = -\frac{1}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} = -\frac{1}{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}} \\ &= -\frac{1}{-1} = 1 \end{aligned}$$

$$(A - BK)^{-1} = \frac{1}{1} \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

Hence

$$u = -\begin{bmatrix} -1 & 1 \end{bmatrix} x + r = x_1 - x_2 + r.$$

(g) There exists L such that $\sigma(A - LC) = \{-1, -1\}$ if and only if $\sigma(A^T - C^T L^T) = \{-1, -1\}$. Set

$$\tilde{A} = A^T = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \tilde{B} = C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \tilde{K} = L^T.$$

Notice that finding $\tilde{k}$ so that $\sigma(\tilde{A} - \tilde{B}\tilde{k}) = \{-1, -1\}$ was solved in (d). Hence $\tilde{K} = [-1 \quad 1]$ and therefore

$$L = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

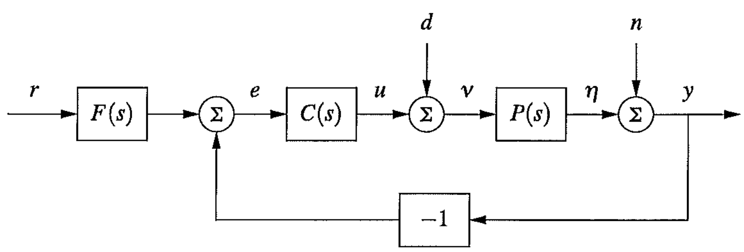

Figure 2: Negative feedback block diagram considered in Problem 4, with $F(s) = 1$.

4. [16pts] You are requested to design a servomechanism (feedback controller) for the angular positioning of a mechanical system described by the equations

$$J\ddot{\theta} = -k\dot{\theta} + u$$

with measured output

$$y = \theta,$$

and where J, k are positive parameters.

(a) [2pts] Given the choice of variables $x_1 = \theta$, $x_2 = \dot{\theta}$, determine the state space representation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

Write the matrices A, B, C, D.

(b) [3pts] Determine the transfer function $P(s)$ of the system.

(c) [3pts] Consider the negative feedback control system in Figure 2, with

$$F(s) = 1, \quad n = 0, \quad d = 0.$$

Let $P(s)$ be as in the previous question and $C(s)$ be the P controller

$$C(s) = k_p.$$

Determine all the values of the gain k_p such that the closed-loop system is asymptotically stable. Note that you can expect these values to depend on the parameters J, k.
Hint If you could not find any transfer function in (b) let $P(s) = \frac{s}{s^2+1}$.

(d) [2pts] Determine the transfer function $G_{yr}(s)$ from the reference r to the output y and the the steady state output response to a step reference input when $n = d = 0$.

(e) [2pts] For this question and the following one, let k_p be a parameter (hence, if you did not determine the values of k_p in the previous question, you can still answer these questions). Determine the transfer function $G_{yn}(s)$ from the noise n to the output y and the steady state output response to a step noise n when $r = d = 0$.

(f) [2pts] Determine the transfer function $G_{yd}(s)$ from the load disturbance d to the output y and the steady state output response to a step load disturbance d when $r = n = 0$.

(g) [2pts] Suppose that the output response to a step reference is required to have a rise time of 0.9 units of time and an overshoot of 0%. What should the coefficients ζ and ω_0 be in the denominator $s^2 + 2\zeta\omega_0 s + \omega_0^2$ of the closed-loop transfer function?

(h) [3pts] **(Bonus)** Consider now a PID controller

$$C(s) = \frac{k_i + k_p s + k_d s^2}{s}.$$

Determine the parameters k_p, k_d, k_i in such a way that the closed-loop system transfer function has the denominator equal to

$$(s^2 + 2\zeta\omega_0 s + \omega_0^2)(s + a)$$

where $a = 5\zeta\omega_0$.

(i) [2pts] **(Bonus)** Determine the steady state output response to a step load disturbance d when the controller is a PID and when $r = n = 0$.

(a)

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -J^{-1}kx_2 + J^{-1}u \\ y &= x_1 \end{cases}$$

$$\begin{aligned} \dot{x} &= Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -J^{-1}k \end{bmatrix} x + \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix} u \\ y &= Cx + Du = \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0u \end{aligned}$$

For each matrix give [0.5 pts]

(b)

$$P(s) = C(sI - A)^{-1}B + D$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s + J^{-1}k \end{bmatrix}^{-1} = \frac{1}{s^2 + J^{-1}ks} \begin{bmatrix} s + J^{-1}k & 1 \\ 0 & s \end{bmatrix} \quad [1pt]$$

$$\begin{aligned} P(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + J^{-1}k & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix} \frac{1}{s^2 + J^{-1}ks} + 0 \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} J^{-1} \\ sJ^{-1} \end{bmatrix} \frac{1}{s^2 + J^{-1}ks} = \frac{J^{-1}}{s^2 + J^{-1}ks} \\ &= \frac{1}{Js^2 + ks} \end{aligned} \quad [1pt]$$

(c) Transfer function of the closed-loop system

$$\frac{L(s)}{1 + L(s)} = \frac{\frac{k_p}{Js^2+ks}}{1 + \frac{k_p}{Js^2+ks}} = \frac{k_p}{Js^2 + ks + k_p} \quad [1pt]$$

k_p must be so that $Js^2 + ks + k_p = 0$ has all roots with strictly negative real part

$$s_{1/2} = \frac{-k \pm \sqrt{k^2 - 4Jk_p}}{2J}$$

Hence $k_p > 0$ [1pt]

Alternative If $P(s) = \frac{s}{s^2+1}$, then $\frac{L(s)}{1+L(s)} = \frac{k_p s}{s^2 + k_p s + 1}$ [1pt]

$$s_{1/2} = \frac{-k_p \pm \sqrt{k_p^2 - 4}}{2}$$

Hence $k_p > 0$ [1pt]

(d)

$$G_{yr}(s) = \frac{k_p}{Js^2 + ks + k_p} \quad [1pt]$$

$$y_{step} = \lim_{s \leftarrow 0} sY(s) = \lim_{s \leftarrow 0} sG_{yr}(s)\frac{r}{s} = r \quad [1pt]$$

(e)

$$G_{yn}(s) = \frac{1}{1+PC} = \frac{Js^2 + ks}{Js^2 + ks + k_p} \quad [1pt]$$

$$y_{noise} = \lim_{s \leftarrow 0} sG_{yn}(s)\frac{\bar{n}}{s} = 0 \quad [1pt]$$

(f)

$$G_{yd}(s) = \frac{P}{1+PC} = \frac{\frac{1}{Js^2+ks}}{1 + \frac{k_p}{Js^2+ks}} = \frac{1}{Js^2 + ks + k_p} \quad [1pt]$$

$$y_{disturbance} = \lim_{s \leftarrow 0} sG_{yd}\frac{\bar{d}}{s} = \frac{\bar{d}}{k_p} \quad [1pt]$$

(g) From Table 6.1 in the textbook, $\zeta = 1$ [0.5 pts], and $\frac{2.7}{\omega_0} = 0.9$, returning $\omega_0 = 3$ [1.5 pts].

(h)

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}, \qquad P(s) = \frac{1}{Js^2 + ks}$$

$$\begin{aligned} \frac{L(s)}{1+L(s)} &= \frac{k_d s^2 + k_p s + k_i}{s(Js^2+ks) + k_d s^2 + k_p s + k_i} \\ &= \frac{k_d s^2 + k_p s + k_i}{Js^3 + (k+k_d)s^2 + k_p s + k_i} \\ &= \frac{1}{J}\frac{k_d s^2 + k_p s + k_i}{s^3 + \frac{k+k_d}{J}s^2 + \frac{k_p}{J}s + \frac{k_i}{J}} \end{aligned}$$

$$\begin{aligned} (s^2 + 2\zeta\omega_0 s + \omega_0^2)(s+a) &= s^3 + as^2 + 2\zeta\omega_0 s^2 + 2a\zeta\omega_0 s + \omega_0^2 s + a\omega_0^2 \\ &= s^3 + (a + 2\zeta\omega_0)s^2 + (2a\zeta\omega_0 + \omega_0^2)s + a\omega_0^2 \end{aligned}$$

$$\left\{ \begin{aligned} \frac{k+k_d}{J} &= a + 2\zeta\omega_0 \\ \frac{k_p}{J} &= 2a\zeta\omega_0 + \omega_0^2 \\ \frac{k_i}{J} &= a\omega_0^2 \end{aligned} \right. \qquad \left\{ \begin{aligned} k_d &= (a + 2\zeta\omega_0)J - k \\ k_p &= J(2a\zeta\omega_0 + \omega_0^2) \\ k_i &= Ja\omega_0^2 \end{aligned} \right.$$

with $a = 5\zeta\omega_0$.

(i) The integral action before the occurrence of the disturbance guarantees that the disturbance eventually has no effect on the output. Hence, $y_{steady} = 0$. Analitically,

$$y_{steady} = \lim_{s \leftarrow 0} sG_{yd}(s)\frac{\bar{d}}{s} = G_{yd}(0)\bar{d} = \left.\frac{s}{Js^3 + (k + k_d)s^2 + k_p s + k_i}\right|_{s=0} \bar{d} = 0$$

5. [15pts] Consider a negative feedback system as in Figure 2, where the process transfer function is

$$P(s) = \frac{1}{(s+1)(s+5)}.$$

(a) [4pts] Design a controller $C(s)$ with as little number of poles as possible such that the closed-loop system has

i. a zero steady state error response to a step input and

ii. a constant e_{steady} steady state error response to a ramp input such that $|e_{steady}| \leq 0.5$.

(b) [2pts] Suppose that the resulting open-loop transfer function $L(s) = C(s)P(s)$ has the Bode diagrams represented in Figure 3. Using the detailed (magnified) Bode diagrams in Figure 4, determine the gain crossover frequency ω_{gc} and the phase $\angle L(i\omega_{gc})$.

(c) [5pts] Using the Bode diagrams represented in Figure 3, qualitatively draw the corresponding Nyquist plot. Determine whether the system is asymptotically stable giving P, N, Z (P the number of poles of $P(s)$ with positive real parts, N the net number of clockwise encirclements of -1, Z the number of poles of the closed-loop system with positive real parts). Explain.

(d) [2pts] Determine the phase margin.

(e) [2pts] Determine the gain margin.

(f) [5pts] **(Bonus)** Consider the Bode diagrams of $S(s) = 1/(1+L(s))$ represented in Figure 5 and 6. Compute the gain and (the approximate) phase of the closed-loop system at the frequency $\omega = 1$. Suppose that the reference signal is $r(t) = \sin(t)$. Compute the steady state error response of the closed-loop system to such a reference signal.

(a)

$$C(s) = \frac{k_p}{s} \quad \text{[2pts]}; W_{er} = \frac{1}{1+L(s)} = \frac{s(s+1)(s+5)}{s^3+6s^2+5s+k_p}$$

$$|e_{steady}| = \left|\lim_{s\to 0} sW_{er}(s)\frac{1}{s^2}\right| = \left|\frac{5}{k_p}\right|; \quad |e_{steady}| \leq 0.5 \quad \text{iff} \quad |k_p| \geq 0 \qquad \text{[1pt]}$$

Since systemin closed-loop is asymptotically stable only for $0 < k_p < 30$, then

$$C(s) = \frac{10}{s} \qquad \text{[1pt]}$$

is an acceptable solution.

(b) $\omega_{gc} \approx 1.2$ rad/sec [1 pt]; $\angle L(i\omega_{gc}) \approx -140°$[1 pt]

(c)

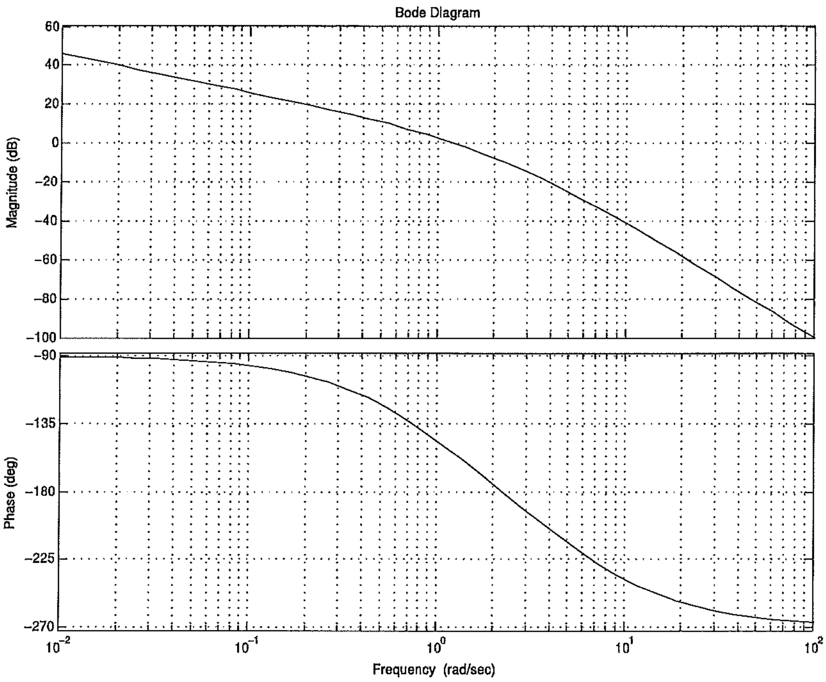

Figure 3: Bode diagram of the closed-loop transfer function $L(s) = C(s)P(s)$.

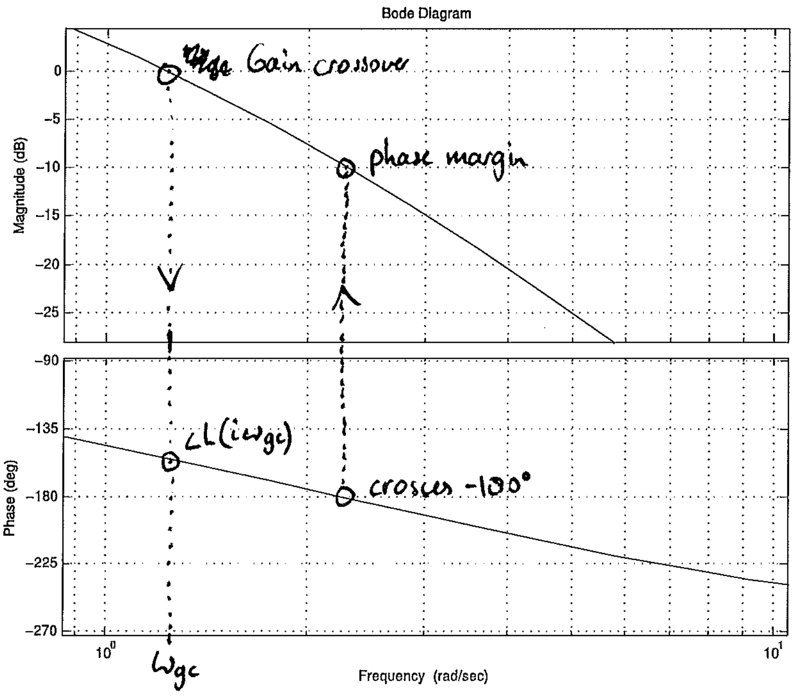

Figure 4: Magnified picture of the Bode diagram of the loop transfer function $L(s) = C(s)P(s)$.

Figure 5: Bode diagram of transfer function $\frac{1}{1+L(s)}$.

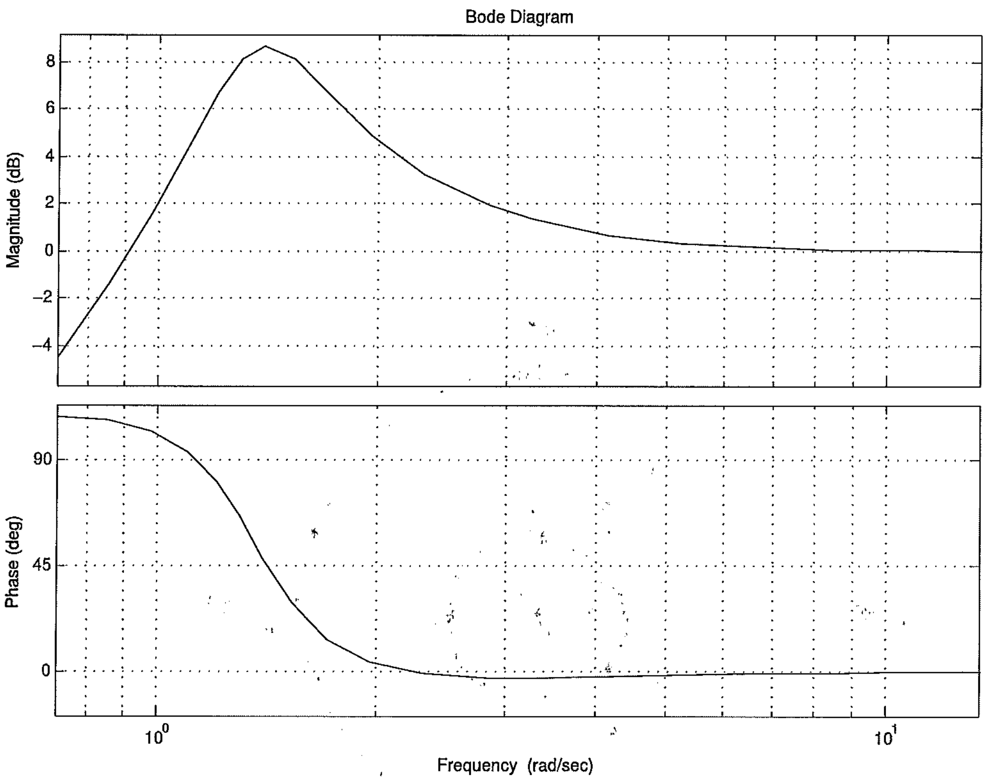

Figure 6: Magnified picture of the Bode diagram of transfer function $\frac{1}{1+L(s)}$.

1. Bode diagram enters at high magnitude, phase -90° (south)
2. For 0dB (unit circle in Nyquist plot) it is at -140°
3. Bode exits at low magnitude, phase -270° (north)
4. Symmetry in R-axis
5. Close the contour 'at infinity.'

[3 pts]

See below for sketch.

$P = 0$, $N = 0$, $Z = 0$. [2 pts]

(d) Phase margin$=180^o + \angle L(i\omega_{gc}) \approx 40^o$

(e) Gain margin $= 10^{9/20} \approx 2.8$
In fact, $\angle L(i\omega_{gc}) = 1 \Leftrightarrow \omega_{gc} \approx 2.2$ rad/sec; $|L(i\omega_{gc})|_{dB} \approx -9$,

$$20\log_{10}|\omega_{gc}| = -9 \qquad |\omega_{gc}| = 10^{-9/20} = \frac{1}{10^{9/20}}$$

(f) $|S(i1)|_{dB} = 2 \Leftrightarrow 20\log_{10}|S(i1)| = 2 \quad |S(i1)| = 10^{1/10} \approx 1.26$ [1.5 pts]

$$\angle S(i1) \approx 112^o = 1.94\text{rad} \qquad [1.5\text{pts}]$$

$$e_{steady}(t) = 1.26\sin(t + 1.94)\text{rad} \qquad [2\text{pts}]$$